

CONJUGATED GRAETZ PROBLEMS—I

GENERAL FORMALISM AND A CLASS OF SOLID-FLUID PROBLEMS

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Abstract—Conjugated Graetz problems arise where two (or more) phases, with at least one phase in fully developed flow, exchange energy or mass across an intervening surface. In these problems the temperature or concentration fields are coupled through the conjugating conditions which express the continuity of fluxes and the rate of transfer. A general formalism is presented first for the analysis of these problems, employing a matrix differential operator with respect to the radial variable and following the decomposition technique of [1-3]. The aforementioned operator is shown to be symmetric in its domain, possessing a denumerable set of eigenvalues and a complete set of eigenvectors. A class of solid-fluid problems, involving the removal of heat from a heated solid cylinder by a surrounding annular-flow fluid, is then discussed in detail; analytical solutions are obtained by expansion in terms of the above eigenvectors. These solid-fluid problems may be viewed as extensions of the extended, one-phase, Graetz problem with prescribed wall flux [2].

1. INTRODUCTION

The Graetz problem for a single fluid stream involves the fluid in rectilinear, fully developed flow through a conduit of constant cross-section exchanging heat (or mass) with the surroundings. The standard variations in this situation consist of a specified wall temperature or wall flux or transfer rate across the wall to the exterior of known ambient temperature. The mathematical formulation leads to an energy differential equation together with boundary conditions at the conduit wall and axis. It is usually appropriate to neglect axial molecular transport in the foregoing problems, which lead to analytical solutions. Low Peclet number flows impose the inclusion of the axial conduction term in the energy equation leading to problems whose solutions have required approximate methods. Recently however, the present authors [1-3] have obtained analytical solutions to the low Peclet number problems based on a new selfadjoint formalism. The objective of the present work is to extend the methodology to conjugated Graetz problems in which two (or more) fluids in co- or counter-current flow exchange energy across an intervening heat transfer surface. In those problems the temperature fields are coupled through interfacial conditions expressing the continuity of fluxes and the rate of heat transfer.

Our efforts [1-3] on the single-stream extended Graetz problems were motivated entirely by the difficulties arising from the inclusion of the axial conduction in the analysis. In dealing with conjugated problems, however, there exist several aspects in the formulation, analysis

and computational features of even the simplest of them which require some fresh perspectives. For example, the solution of some parabolic-elliptic, liquid-solid problems may be readily obtained, without resorting to the solution of equivalent integral equations. In other problems, the effect of a solid phase separating two fluid phases may be incorporated into the analysis very conveniently, as we shall see in the general formulation and analysis that follow. For the counter-current problems we provide a convenient formalism, define a suitable Hilbert space and derive the necessary expansion theorem.

The difficulties in the treatment of the conjugated Graetz problems are discussed in some detail in [4] and may be briefly summarized as follows:

(1) The integration of the eigenvalue problems in annular spaces and/or with non-parabolic velocity profiles cannot be accomplished in a straight-forward manner. In the past, this led to over-simplifications of the problems or to the use of entirely numerical integration techniques (Runge-Kutta), thus limiting the applications of the problems and the computational efficiency of their solutions.

(2) For the parabolic counter-current problem the usual inner product with the velocity expression as the weight function is not available, since the velocity changes sign over the radial interval.

(3) Conjugated problems with a flow phase described by an elliptic equation (i.e. when axial conduction is included in the analysis), inherit all the difficulties of the single stream problems which have been discussed in [1-3].

(4) Conjugated problems involving at least one solid phase and one fluid phase, may be treated as elliptic

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problems deriving from the solid phase, although axial conduction may be insignificant for the fluid phase. This fact has not been recognized before, so that these problems were treated through their equivalent integral equations.

(5) Problems involving thin solid phases, the resistance of which is accounted for through a simple relationship, introduce discontinuities in the radial interval, thus requiring a modified inner product to yield a self-adjoint formalism.

The present analysis overcomes in principle all of the above difficulties by employing a new operator formalism. Thus it employs an inner product that does not contain the velocity expression, so that the analysis is possible whether the velocity is negative (counter-current problems) or zero (for solid phases).

Among problems involving essentially fluid phases only, some have been treated through their differential formulation [5-15] and others through the formulation of equivalent integral equations [16-18].

The applications of conjugated problems are numerous, covering many areas of chemical engineering and of other disciplines. Thus, we have, for example, conjugated problems involving liquid-solid-liquid or liquid-solid-gas interfaces, with applications in heat exchangers and mass separation processes with or without chemical reaction. Metallurgical systems involve mass transfer of different species across gas-liquid, liquid-liquid or solid-liquid interfaces. Wetted-wall absorbers and falling-film reactors involve gas-liquid interfaces. References for all the above and further applications may be found in [16-18].

In this paper we shall present the formalism for the most general problem, involving any dual combination of fluid and solid phases. Fluid phases are assumed to be separated by a thin solid phase (offering interfacial resistance). Then, we shall solve a class of solid-fluid problems which admit very simple analytical solutions. The solution of the problems is based on the methodology of our earlier papers [1-3], which (methodology) decomposes the energy differential equation into a system of first order differential equations such that a selfadjoint formalism is obtained.

The present paper also generalizes the papers of Ramkrishna and Amundson on the solution of transport problems with oblique and mixed derivative boundary conditions [19,20], in that it accounts for transverse variations in the temperature or concentration field and axial transport in the fluid.

2. GENERAL FORMALISM

Consider two fluids with fully developed flows exchanging heat as in Fig. 1. One fluid is flowing in the tubular space of the double pipe, while the other fluid is flowing in the surrounding annular space co-currently or counter-currently. If one of the two velocities is zero,

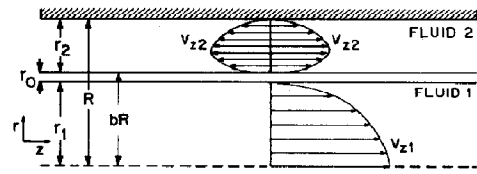


Fig. 1. Sketch of the physical situation described by the general conjugated Graetz problem.

then we have the case of a solid-fluid conjugated problem. The thickness of the wall that separates the two media, r_0 , is small but finite. The outer fluid is exchanging energy, through the outer wall, with a third fluid of constant temperature. The physical properties for each fluid are assumed to be constant and we neglect any viscous dissipation.† The specification of the axial domain and boundary conditions is deferred to a later stage. Assuming axisymmetry, we may write the energy equations and the boundary conditions for the two fluids, in compact form as follows,

$$-\frac{1}{r} \frac{\partial}{\partial r} \left[k^*(r) r \frac{\partial T}{\partial r} \right] - k^*(r) \frac{\partial^2 T}{\partial z^2} + \rho(r) c_p(r) v_z(r) \frac{\partial T}{\partial z} = 0 \quad \begin{array}{l} 0 < r < r_1 \\ r_0 + r_1 < r < R \end{array} \quad (1)$$

$$\frac{\partial T}{\partial r} = 0 \quad r = 0, \quad \text{all } z \quad (2)$$

$$-k_2 \frac{\partial T}{\partial r} \Big|_{r=R} = h_e [T(z, R) - T_0], \quad \text{all } z \quad (3)$$

$$-k_1 \frac{\partial T}{\partial r} \Big|_{r=r_1} = h [T(z, r_1) - T(z, r_1 + r_0)] \quad \text{all } z \quad (4)$$

$$-k_1 \frac{\partial T}{\partial r} \Big|_{r=r_1} = -k_2 \frac{r_1 + r_0}{r_1} \cdot \frac{\partial T}{\partial r} \Big|_{r=r_1 + r_0}, \quad \text{all } z \quad (5)$$

with

$$k^*(r) = \begin{cases} k_1 \\ k_2 \end{cases} \quad \rho(r) = \begin{cases} \rho_1 \\ \rho_2 \end{cases} \quad c_p(r) = \begin{cases} c_{p1} \\ c_{p2} \end{cases}$$

$$v_z(r) = \begin{cases} v_{z1}(r) \\ v_{z2}(r) \end{cases}$$

where in the above four equalities, the quantities with subscript 1 correspond to the interval $0 < r < r_1$, while those with subscript 2 to the interval $r_1 + r_0 < r < R$. With laminar flows for both fluids we have,

$$v_{z1}(r) = V_1 \left(1 - \frac{r^2}{r_1^2} \right) \quad (6)$$

$$v_{z2}(r) = \delta V_2 \left(1 - \left(\frac{r}{R} \right)^2 + a \ln \frac{r}{R} \right) \quad (7)$$

where

$$a = \frac{b^2 - 1}{\ln b} \quad (8)$$

†This assumption is made only for reasons of convenience. The herein presented formalism can accommodate viscous dissipation as long as the assumption of constant physical properties is upheld.

and $\delta = +1$ for the co-current and -1 for the counter-current problems, while constants V_1 and V_2 may be found in Bird *et al.* ([21], pp. 46 and 53, respectively). In writing eqn (4) we assume that since the thickness, r_0 , of the separating wall is small, axial conduction in this wall may be neglected and its resistance to heat transfer may be described by the heat transfer coefficient h . Similarly for the outer wall (eqn 3). Equation (5) expresses the continuity of fluxes from the two sides of the inner wall.

In dimensionless form we may write the problem as,

$$M\Theta = -\frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \Theta}{\partial \eta} \right) - \frac{1}{Pe^2} \frac{\partial^2 \Theta}{\partial \zeta^2} + v(\eta) \frac{\partial \Theta}{\partial \zeta} = 0 \quad \begin{matrix} 0 < \eta < b-d \\ b < \eta < 1 \end{matrix} \quad (9)$$

$$\frac{\partial \Theta}{\partial \eta}(\zeta, 0) = 0 \quad (10)$$

$$-\frac{\partial \Theta}{\partial \eta}(\zeta, 1) = B_* \Theta(\zeta, 1) \quad (11)$$

$$-\frac{\partial \Theta}{\partial \eta}(\zeta, b-d) = B[\Theta(\zeta, b-d) - \Theta(\zeta, b)] \quad (12)$$

$$\frac{\partial \Theta}{\partial \eta}(\zeta, b-d) = k \frac{\partial \Theta}{\partial \eta}(\zeta, b) \quad (13)$$

with

$$v(\eta) = \begin{cases} \frac{1}{A} \left(1 - \frac{\eta^2}{(b-d)^2} \right) = v_1(\eta) & 0 < \eta < b-d \\ \delta(1 - \eta^2 + a \ln \eta) = v_2(\eta) & b < \eta < 1. \end{cases} \quad \begin{matrix} (14a) \\ (14b) \end{matrix}$$

Notice that the whole dedimensionalization process was based on the properties of fluid 2, so that the Peclet number in (9) is the Peclet number of fluid 2. This was found to have some computational advantages in the integration of the resulting eigenvalue problem.

In order to facilitate the decomposition of eqn (9) into a pair of first order partial differential equations, we define the axial-energy-flow function[1-3] separately for each fluid as follows,

$$S_1(z, r) = \int_0^r \left[-k_1 \frac{\partial T}{\partial z} + \rho_1 c_{p1} v_{z1} \right] 2\pi r' dr', \quad 0 < r < r_1 \quad (15)$$

$$S_2(z, r) = S_1(z, r_1) + \int_{r_1+r_0}^r \left[-k_2 \frac{\partial T}{\partial z} + \rho_2 c_{p2} v_{z2} T \right] 2\pi r' dr', \quad r_1 + r_0 < r < R. \quad (16)$$

Now, we may write as in the single stream problems [1-4] and in view of (5),

$$\frac{\partial S_1}{\partial z} = -2\pi r \left[-k_1 \frac{\partial T}{\partial r} \right] \quad 0 < r < r_1 \quad (17)$$

$$\frac{\partial S_1}{\partial r} = \left[-k \frac{\partial T}{\partial z} + \rho_1 c_{p1} v_{z1} T \right] 2\pi r \quad 0 < r < r_1 \quad (18)$$

and

$$\frac{\partial S_2}{\partial z} = -2\pi r \left[-k_2 \frac{\partial T}{\partial r} \right] \quad r_1 + r_0 < r < R \quad (19)$$

$$\frac{\partial S_2}{\partial r} = \left[-k_2 \frac{\partial T}{\partial z} + \rho_2 c_{p2} v_{z2} T \right] 2\pi r \quad r_1 + r_0 < r < R. \quad (20)$$

It should be noted that (16) and (19) imply (5). Then, we define

$$S(z, r) \equiv \begin{cases} S_1(z, r) & 0 < r < r_1 \\ S_2(z, r) & r_1 + r_0 < r < R. \end{cases} \quad (21)$$

Elimination of S_1 between (17) and (18) and S_2 between (19) and (20) produces eqn (1), so that the system of eqns (17)-(20) is equivalent to eqn (1). Further, we obviously have

$$S(z, 0) = 0. \quad (22)$$

To switch to dimensionless variables we define,

$$\Sigma_1(\zeta, \eta) \equiv \frac{\gamma}{\pi R^2 \rho_2 c_{p2} V_2 T_D} \left[S_1(z, r) - \int_0^r \rho_1 c_{p1} v_{z1} T_2 2\pi r' dr' \right] \quad \begin{matrix} 0 < \eta < b-d \\ 0 < r < r_1 \end{matrix} \quad (23)$$

$$\Sigma_2(\zeta, \eta) \equiv \frac{1}{\pi R^2 \rho_2 c_{p2} V_2 T_D} \left[S_2(z, r) - \int_0^{r_1} \rho_1 c_{p1} v_{z1} T_2 2\pi r' dr' - \int_{r_1+r_0}^r \rho_2 c_{p2} v_{z2} T_2 2\pi r' dr' \right] \quad \begin{matrix} b < \eta < 1 \\ r_1 + r_0 < r < R. \end{matrix} \quad (24)$$

Defining now

$$\Sigma(\zeta, \eta) = \begin{cases} \Sigma_1(\zeta, \eta) & 0 < \eta < b-d \\ \Sigma_2(\zeta, \eta) & b < \eta < 1 \end{cases} \quad (25)$$

from (15), (16), (23) and (24) we obtain

$$\Sigma(\zeta, \eta) = \int_0^\eta \left[-\frac{1}{Pe^2} \frac{\partial \Theta}{\partial \zeta} + v \Theta \right] 2\eta' d\eta' \quad 0 < \eta < b-d \quad (26)$$

$$\Sigma(\zeta, \eta) = \frac{1}{\gamma} \Sigma(\zeta, b-d) + \int_b^\eta \left[-\frac{1}{Pe^2} \frac{\partial \Theta}{\partial \zeta} + v \Theta \right] 2\eta' d\eta' \quad b < \eta < 1 \quad (27)$$

where

$$\gamma \equiv \frac{k_2}{k_1}. \quad (28)$$

Equations (17)-(20) with a slight rearrangement of (18) and (20), may be written in a compact dimensionless form as,

$$\frac{\partial \Theta}{\partial \zeta} = Pe^2 v(\eta) \Theta - \frac{Pe^2}{2\eta} \frac{\partial \Sigma}{\partial \eta} \quad (29)$$

$$\frac{\partial \Sigma}{\partial \zeta} = 2\eta \frac{\partial \Theta}{\partial \eta}. \quad (30)$$

Note that elimination of Σ from (29) and (30) produces

(9). Further, from (22), (23) and (27) we obtain,

$$\Sigma(\zeta, 0) = 0 \tag{31}$$

$$\gamma \Sigma(\zeta, b) = \Sigma(\zeta, b - d). \tag{32}$$

From (30) and (12), we have,

$$\begin{aligned} \frac{\partial \Sigma}{\partial \zeta}(\zeta, b - d) &= 2(b - d) \frac{\partial \Theta}{\partial \eta}(\zeta, b - d) \\ &= -2(b - d) B [\Theta(\zeta, b - d) - \Theta(\zeta, b)] \end{aligned} \tag{33}$$

and also from (30) and (11),

$$\frac{\partial \Sigma}{\partial \zeta}(\zeta, 1) = 2 \frac{\partial \Theta}{\partial \eta}(\zeta, 1) = -2B_e \Theta(\zeta, 1). \tag{34}$$

In view of (33) and (34), we may write eqns (29), (30), (11) and (12) in the following compact form of matrix notation,

$$\frac{\partial}{\partial \zeta} F(\zeta, \eta) n = L F(\zeta, \eta) \tag{35}$$

where

$$L = \begin{bmatrix} Pe^2 v(\eta) & -\frac{Pe^2}{2\eta} \frac{\partial}{\partial \eta} & 0 & 0 \\ 2\eta \frac{\partial}{\partial \eta} & 0 & 0 & 0 \\ -2B(b - d) \left[\lim_{\eta \rightarrow b-d} - \lim_{\eta \rightarrow b} \right] & 0 & 0 & 0 \\ -2B_e \lim_{\eta \rightarrow 1} & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

and

$$F(\zeta, \eta) = [\Theta(\zeta, \eta), \Sigma(\zeta, \eta), \Sigma(\zeta, b - d), \Sigma(\zeta, 1)]^T. \tag{37}$$

Now the problem may be recast as given by eqn (35) subject to boundary conditions (31), (32) and some boundary conditions w.r.t. ζ that must be specified for the particular problem.

In view of (31) and (32), if we define the inner product between two vectors

$$\phi = [\phi_1(\eta), \phi_2(\eta), \phi_3, \phi_4]^T \tag{38}$$

and

$$\psi = [\psi_1(\eta), \psi_2(\eta), \psi_3, \psi_4]^T \tag{39}$$

as

$$\begin{aligned} \langle \phi, \psi \rangle &= \int_0^{b-d} X(\phi, \psi) d\eta \\ &+ \gamma \int_b^1 X(\phi, \psi) d\eta + \frac{\phi_3 \psi_3}{(b-d)B} + \gamma \frac{\phi_4 \psi_4}{B_e} \end{aligned} \tag{40}$$

where

$$X(\phi, \psi) = \frac{4}{Pe^2} \phi_1(\eta) \psi_1(\eta) \eta + \phi_2(\eta) \psi_2(\eta) \frac{1}{\eta} \tag{41}$$

then L is symmetric in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{R} \oplus \mathcal{R}$ with domain

$$\begin{aligned} D(L) &= \{ \phi \in \mathcal{H} : L\phi \text{ (exists and)} \in \mathcal{H}, \phi_3 = \phi_2(b - d), \\ &\phi_4 = \phi_2(1), \phi_2(0) = 0, \phi_2(b - d) = \gamma \phi_2(b) \} \end{aligned} \tag{42}$$

as shown in Appendix 1. Hilbert space \mathcal{H}_1 and \mathcal{H}_2 are clearly defined also in Appendix 1. Unlike all single-stream Graetz problems, the η domain is not a single continuous interval, but rather the union of the two continuous intervals $[0, b - d]$ and $[b, 1]$. Of course, all the linear operator theory is valid with this kind of domains. Thus, we obtain the selfadjoint (or Sturm-Liouville) eigenvalue problem,

$$L\phi_i = \lambda_i \phi_i \tag{43}$$

from which we may write, analytically,

$$Pe^2 v(\eta) \phi_{i1}(\eta) - \frac{Pe^2}{2\eta} \phi'_{i2}(\eta) = \lambda_i \phi_{i1}(\eta) \tag{44}$$

$$2\eta \phi'_{i1}(\eta) = \lambda_i \phi_{i2}(\eta) \tag{45}$$

$$-2B(b - d) [\phi_{i1}(b - d) - \phi_{i1}(b)] = \lambda_i \phi_{i2}(b - d) \tag{46}$$

$$-2B_e \phi_{i1}(1) = \lambda_i \phi_{i2}(1). \tag{47}$$

From (44)–(47) and (42) we may now obtain the problem,

$$\frac{1}{\eta} (\eta \phi'_{i1})' + \left[\frac{\lambda_i}{Pe^2} - v(\eta) \right] \lambda_i \phi_{i1} = 0 \tag{48}$$

$$\phi'_{i1}(0) = 0 \tag{49}$$

$$\phi'_{i1}(b - d) = k \phi'_{i1}(b) \tag{50}$$

$$-\phi'_{i1}(b - d) = B [\phi_{i1}(b - d) - \phi_{i1}(b)] \tag{51}$$

$$\phi'_{i1}(1) + B_e \phi_{i1}(1) = 0. \tag{52}$$

The gap in the η domain is a result of including in the analysis the effect of the separating inner wall. Although the effect of the wall is approximated with relation (12), we cannot ignore the effect of its thickness on the velocity field and thus on the temperature field. This finite thickness of the inner wall necessitates also the usage of the inner product (40).

Operator L is neither positive definite nor negative definite. It has both positive eigenvalues $\{\lambda_i^+\}$ with corresponding eigenvectors $\{\phi_i^+\}$ and negative eigenvalues $\{\lambda_i^-\}$ with eigenvector $\{\phi_i^-\}$. The two sets of eigenvectors, normalized according to inner product (40), together constitute an orthonormal basis in \mathcal{H} . For the co-current or the solid-fluid problems the positive spectrum disappears for $Pe \rightarrow \infty$ as with the single stream problems, for, one may show that in this case the operator becomes asymptotically negative definite. For the counter-current problems however the positive spectrum is retained even for $Pe \rightarrow \infty$, which is because of the fact that the velocity expression changes sign over the η domain.

The following expansion theorem results from the eigenvalue problem (42)–(43),

$$f = \sum_{j=1}^{\infty} \left[\frac{\langle f, \phi_j^+ \rangle}{\|\phi_j^+\|^2} \phi_j^+ + \frac{\langle f, \phi_j^- \rangle}{\|\phi_j^-\|^2} \phi_j^- \right] = \sum_{j=1}^{\infty} \frac{\langle f, \phi_j \rangle}{\|\phi_j\|^2} \phi_j \quad (53)$$

for every vector $f \in \mathcal{X}$.

3. SOLID-FLUID PROBLEMS

Solid-fluid conjugated Graetz problems involve heat or mass transfer between a stationary phase (a true solid, a fluid or a gel) and a flowing phase. By virtue of the thermal or mass diffusion in the “solid” phase, they are elliptic-parabolic problems, if axial conduction or diffusion may be neglected in the flowing phase, or elliptic-elliptic problems if not. For the currently employed formalism however, they are elliptic problems in either case.

Solid-fluid conjugated Graetz problems find a wide range of applications both in heat transfer [18, 22, 23] and mass transfer [24, 25]. These problems have been treated in the past essentially through their equivalent integral-equation formulation. The present methodology avoids the solution of integral equations and provides a strictly analytical solution.

3.1. A problem with heat transfer through the outer wall

Consider the physical situation depicted in Fig. 2. A viscous fluid is flowing in an annular space around a solid cylinder to remove the heat produced by the heat source $F(z, r)$ occupying part of the cylinder. The fluid is also exchanging heat through the outer wall with the surrounding environment of constant temperature T_2 . The temperature with which the fluid is entering the annular at $-\infty$ is also T_2 . This problem for example may physically represent removal of heat generated by a nuclear rod. The solid cylinder is for simplicity assumed to be of uniform thermal conductivity. The length of the heat-source section of the cylinder is l_1 . If the heat source is axisymmetric, the mathematical problem, in dimensionless form, may be written as,

$$M\Theta = f(\zeta, \eta) \quad \begin{matrix} 0 < \eta < b-d \\ b < \eta < 1 \end{matrix} \quad (54)$$

subject to boundary conditions (10)–(13). Differential operator M is defined by eqn (9), while $v(\eta)$ is given by (14) with $A = \infty$, i.e. with $v_1(\eta) = 0$, and $\delta = 1$. The dimensionless source function $f(\zeta, \eta) = (R^2 F(z, r))/(k, T_2)$. If we go through the decomposition process of the energy eqn (54) as we did in Section 2, we can show that the problem may be written, utilizing operator L of eqn (36), as follows,

$$\frac{\partial}{\partial \zeta} F(\zeta, \eta) = LF(\zeta, \eta) + E(\zeta, \eta) \quad (55)$$

where

$$E = [0, g(\zeta, \eta), g(\zeta, b-d), g(\zeta, 1)]^T \quad (56)$$

$$g(\zeta, \eta) =$$

$$\begin{cases} \int_0^{\eta} f(\zeta, r) 2r dr & 0 < \eta < b-d, \quad 0 < \zeta < Z_1 & (57a) \\ \frac{1}{\gamma} \int_0^{b-d} f(\zeta, r) 2r dr & b < \eta < 1, \quad 0 < \zeta < Z_1 & (57b) \\ 0 & \zeta \notin [0, Z_1] & (57c) \end{cases}$$

subject of course to boundary conditions (31) and (32). Note incidentally that the boundary conditions w.r.t. to ζ at $-\infty$ and ∞

$$\Theta(-\infty, \eta) = \Theta(\infty, \eta) = 0 \quad (58)$$

will be found to be satisfied automatically by the forthcoming solution by simply requiring that the solution is finite at $-\infty$ and ∞ .

The solution of the problem is sought in the form of eqn (53) and thus we must determine the inner products of the solution vector with the eigenvectors of L . Since the solution vector, F , for every ζ belongs to the domain of L , $D(L)$, from (55) we have,

$$\frac{d}{d\zeta} \langle F, \phi_j \rangle = \lambda_j \langle F, \phi_j \rangle + \langle E, \phi_j \rangle = \lambda_j \langle F, \phi_j \rangle + h_j(\zeta) \quad (59)$$

from which we obtain, separately for the positive and negative eigenvalues,

$$\langle F, \phi_j^+ \rangle = C_j^+ e^{\lambda_j^+ \zeta} - \int_{\zeta}^{\infty} h_j^+(\zeta') e^{\lambda_j^+(\zeta - \zeta')} d\zeta' \quad (60)$$

and

$$\langle F, \phi_j^- \rangle = C_j^- e^{\lambda_j^- \zeta} + \int_{-\infty}^{\zeta} h_j^-(\zeta') e^{\lambda_j^-(\zeta - \zeta')} d\zeta' \quad (61)$$

Since the solution must be finite at both $-\infty$ and $+\infty$, we have,

$$C_j^+ = C_j^- = 0. \quad (62)$$

At this stage it is necessary to specify the heat-source function $F(z, r)$ or $f(\zeta, \eta)$. We take $F(z, r)$ to be of uniform magnitude $F = T_2 k_1 / R^2$ and thus

$$f(\zeta, \eta) = \begin{cases} 1 & 0 < \zeta < Z_1, \quad 0 < \eta < b-d & (63a) \\ 0 & \text{for all other values of } \zeta \text{ and/or } \eta & (63b) \end{cases}$$

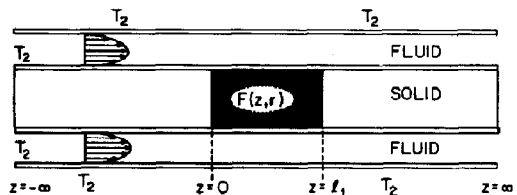


Fig. 2. Sketch of the conjugated solid-fluid Graetz problem discussed in Section 3.1.

where $Z_1 = l_1/(RPe)$. Then, from (57) we have,

$$g(\zeta, \eta) = \begin{cases} \eta^2 & \zeta \in [0, Z_1] \text{ and } \eta \in [0, b-d] \\ (b-d)^2/\gamma & \zeta \in [0, Z_1] \text{ and } \eta \in [b, 1] \\ 0 & \zeta \notin [0, Z_1] \end{cases} \quad \begin{matrix} (64a) \\ (64b) \\ (64c) \end{matrix}$$

Hence,

$$\mathbf{E}(\zeta, \eta) = \begin{cases} \left[0, g(\zeta, \eta), (b-d)^2, \frac{(b-d)^2}{\gamma} \right]^T & \zeta \in [0, Z_1] \\ 0 & \zeta \notin [0, Z_1]. \end{cases} \quad \begin{matrix} (65a) \\ (65b) \end{matrix}$$

Thus, by virtue of (59) and (45)–(47) we may eventually obtain,

$$h_j(\zeta) = \begin{cases} \frac{2}{\lambda_j} \int_0^{b-d} \phi_{j1}(\eta) \eta^2 d\eta \\ - (b-d)^2 \cdot \phi_{j1}(b-d) \\ 0 \end{cases} \equiv m_j, \zeta \in [0, Z_1] \quad \begin{matrix} (66a) \\ (66b) \end{matrix}$$

from which and in view of (60), (61) and (53), the solution may finally be written in the form,

$$\Theta(\zeta, \eta) = \sum_{j=1}^{\infty} \frac{m_j^+ (e^{-\lambda_j^+ \zeta} - 1)}{\lambda_j^+ \|\phi_j^+\|^2} \phi_{j1}(\eta) e^{\lambda_j^+ \zeta} \quad \zeta \leq 0 \quad (67a)$$

$$\Theta(\zeta, \eta) = - \sum_{j=1}^{\infty} \frac{m_j \phi_{j1}(\eta)}{\lambda_j \|\phi_j\|^2} + \sum_{j=1}^{\infty} \frac{m_j^+ e^{-\lambda_j^+ \zeta}}{\lambda_j^+ \|\phi_j^+\|^2} \phi_{j1}(\eta) e^{\lambda_j^+ \zeta} + \sum_{j=1}^{\infty} \frac{m_j^-}{\lambda_j^- \|\phi_j^-\|^2} \phi_{j1}(\eta) e^{\lambda_j^- \zeta} \quad 0 \leq \zeta \leq Z_1 \quad (67b)$$

$$\Theta(\zeta, \eta) = \sum_{j=1}^{\infty} \frac{m_j^- (e^{-\lambda_j^- \zeta} - 1)}{\lambda_j^- \|\phi_j^-\|^2} \phi_{j1}(\eta) e^{\lambda_j^- \zeta} \quad Z_1 \leq \zeta < \infty. \quad (67c)$$

The first series in the righthand side of (67b) is shown in Appendix 2 to converge to $\Psi(\eta)$ given by,

$$\Psi(\eta) = \begin{cases} -\frac{\eta^2}{4} + \frac{b-d}{2} \left[\frac{1}{B} + \frac{b-d}{2} + \frac{b}{k} \left(\frac{1}{B_e} - \ln b \right) \right] & \eta \in [0, b-d] \\ \frac{(b-d)b}{2k} \left(\frac{1}{B_e} - \ln \eta \right) & \eta \in [b, 1]. \end{cases} \quad \begin{matrix} (68a) \\ (68b) \end{matrix}$$

In fact, $\Psi(\eta)$ represents the thermally developed temperature profile in the solid and the fluid for a semi-infinitely long heat source, as it can be readily seen from eqn (67b) by taking the limit for $Z_1 \rightarrow \infty$. For the other limiting case of $Z_1 = 0$, notice that all three expressions (67a)–(67c) give $\Theta(\zeta, \eta) = 0$, as expected.

3.2 A problem with a thermally insulated outer wall

Next we shall consider the problem with the outer wall thermally insulated. A simplified form of this problem (flat velocity profile and no radial temperature variation in

the fluid phase) has been recently solved by Ramkrishna *et al.* [19]. The problem is described again by eqn (54) subject to (10)–(13) with $B_e = 0$. Equation (14) applied again with $A = \infty$ and $\delta = 1$. Equations (15)–(29) also remain valid while (30) must be replaced by,

$$\frac{\partial \Sigma}{\partial \zeta} = \begin{cases} 2\eta \frac{\partial \Theta}{\partial \eta} + \int_0^{\eta} 2\eta' f(\zeta, \eta') d\eta' & \eta \in [0, b-d] \\ 2\eta \frac{\partial \Theta}{\partial \eta} + \frac{1}{\gamma} \int_0^{b-d} 2rf(\zeta, r) dr & \eta \in [b, 1]. \end{cases} \quad \begin{matrix} (69a) \\ (69b) \end{matrix}$$

Equations (31)–(33) also remain the same, while (34) must be replaced as follows. An energy balance for the total cross section of the system between z and $z + dz$ yields,

$$\frac{dS}{dz}(z, R) = \begin{cases} 0 & z < 0 \text{ or } z > l_1 \\ 2\pi \int_0^{r_1} F(z, r) r dr = \pi Fr_1^2 & 0 < z < l_1 \end{cases} \quad \begin{matrix} (70a) \\ (70b) \end{matrix}$$

and thus integrating we obtain,

$$\begin{matrix} S(z, R) = S(-\infty, R) & z < 0 & (71a) \\ S(z, R) = S(-\infty, R) + \pi Fr_1^2 z & 0 < z < l_1 & (71b) \\ S(z, R) = S(-\infty, R) + \pi Fr_1^2 l_1 & z > l_1. & (71c) \end{matrix}$$

In dimensionless form the above equations become,

$$\Sigma(\zeta, 1) = \begin{cases} 0 & \zeta < 0 \\ \frac{(b-d)^2}{\gamma} \zeta & 0 < \zeta < Z_1 \\ \frac{(b-d)^2}{\gamma} Z_1 & \zeta > Z_1. \end{cases} \quad \begin{matrix} (72a) \\ (72b) \\ (72c) \end{matrix}$$

In view of the above, the problem may be written in the following compact form,

$$\frac{\partial \mathbf{F}_1}{\partial \zeta}(\zeta, \eta) = \mathbf{L}_1 \mathbf{F}_1(\zeta, \eta) + \mathbf{E}_1(\zeta, \eta) \quad (73)$$

subject to (31), (32) and (72). \mathbf{L}_1 is obtained by deleting the fourth row and column from \mathbf{L} , while \mathbf{F}_1 and \mathbf{E}_1 are obtained by deleting the fourth elements of \mathbf{F} and \mathbf{E} respectively. The vector elements of the modified Hilbert space have three components now, and the new inner product is derived from (40) by deleting the fourth term. The appropriate domain for \mathbf{L}_1 is obtained from (42) with the additional condition

$$\phi_2(1) = 0 \quad (74)$$

which derives from (72). Now of course, in view of (72) and (74), the solution vector \mathbf{F}_1 will not belong to the domain of \mathbf{L}_1 ; such inhomogeneities may be handled however as in [2]. The solution, is finally given by,

$$\Theta(\zeta, \eta) = \sum_{j=1}^{\infty} B_j^+ (1 - e^{-\lambda_j^+ \zeta}) \phi_{j1}^+(\eta) e^{\lambda_j^+ \zeta} \quad \zeta \leq 0 \quad (75a)$$

$$\Theta(\zeta, \eta) = A_1 \zeta + \Phi(\eta) - \sum_{j=1}^{\infty} B_j^+ e^{-\lambda_j^+ \zeta} \phi_{j1}^+(\eta) e^{\lambda_j^+ \zeta} - \sum_{j=1}^{\infty} B_j^- \phi_{j1}^-(\eta) e^{\lambda_j^- \zeta} \quad 0 \leq \zeta \leq Z_1 \quad (75b)$$

$$\Theta(\zeta, \eta) = A_1 Z_1 - \sum_{j=1}^{\infty} B_j^-(1 - e^{-\lambda_j^- Z_1}) \phi_{j1}^-(\eta) e^{\lambda_j^- \zeta} \quad \zeta \geq Z_1 \quad (75c)$$

where

$$B_j \equiv -m_j / (\lambda_j \|\phi_j\|^2) \quad (76)$$

and $A_1, \Phi(\eta)$ are defined as follows and shown to be given by,

$$A_1 = (b-d)^2 \sum_{j=1}^{\infty} \frac{2\phi_{j1}(1)\phi_{j1}(\eta)}{\lambda_j \|\phi_j\|^2} = \frac{2(b-d)^2}{\gamma(1-b^4 + (b^2-1)^2/\ln b)} \quad (77)$$

$$\Phi(\eta) \equiv \sum_{j=1}^{\infty} B_j \phi_{j1}(\eta) =$$

$$\begin{cases} -\frac{\eta^2}{4} + K_1 + \frac{b-d}{2} \left(\frac{b-d}{2} + \frac{1}{B} \right) & (78a) \\ K_1 - A_1 \int_b^{\eta} \frac{d\eta'}{\eta'} \int_{\eta'}^1 v(r)r dr & (78b) \end{cases}$$

with

$$K_1 = \frac{2\gamma A_1^2}{(b-d)^2} \left\{ \frac{1}{Pe^2} \left[\frac{(b-d)^2}{\gamma} + (1-b)^2 \right] + K_2 \right\} \quad (79)$$

$$K_2 = \int_b^1 v(\eta)\eta d\eta \int_b^{\eta} \frac{d\eta'}{\eta'} v(r)r dr \quad (80)$$

The expressions for A_1 and $\Phi(\eta)$ in (77) and (78) are derived following a procedure similar to the one used by Papoutsakis *et al.* [2]. Note from (75b) that $A_1 \zeta + \Phi(\eta)$ represents the thermally developed temperature profile achieved with a semi-infinite heat source ($Z_1 = \infty$). It is to be noted that although the defining expressions in (77) and (78) are valid with any velocity profile, the derived expressions in (77) and (78) correspond to the laminar profile (14b). Also, in deriving solution (75), the simple heat-source function (63) has been used. The solutions with other prescribed velocity profiles and/or heat-source functions may be derived analogously. It is understood also that in all the above expressions the eigenvalues are different from those of the problem of Section 3.1 although for simplicity we use the same symbols (λ_j, ϕ_j, m_j , etc.) for the corresponding quantities.

Definitions, expressions and relevant computational details for the bulk fluid temperature, local and asymptotic Nusselt numbers for both the problems of Sections 3.1 and 3.2 may be found in [4].

4. COMPUTATIONAL ASPECTS

Computations have been performed on a CDC 6600 computer for the problem of Section 3.1, i.e. with the Robin boundary condition for the outer wall. The following values for the dimensionless parameters have

been used: $b = 0.65, d = 0.05, B = 20, B_c = 1, k = 1$ and various values for Z_1 and Pe .

For the laminar flow considered here the integration of the eigenvalue problem (48)–(51) is performed using a series solution for the annular space (see [4] for details) and the Bessel functions of the zeroth order for the solid cylinder. For any given velocity profile, a series solution or a numerical technique may be used. This integration technique is indeed most efficient. Thus 14 eigenvalues may be computed within 2–3 sec, 8 eigenvalues within 1–1.5 sec and 6 eigenvalues within fractions of a second, all with an accuracy of at least 10 significant figures. However, only 4–6 eigenvalues are required for the final computations of this problem. Similarly 14 expansion coefficients ($= m_j / (\lambda_j \|\phi_j\|^2)$) for solution (67) were computed within 1 sec and 6–8 coefficients within fractions of a second. The final computation of the temperature at 210 points, typically utilized for any of the Figs. 3–6, required 1–1.5 sec, with an accuracy of better than 0.05%. Further computational details may be found in [4].

5. RESULTS AND DISCUSSION

A particularly interesting feature of the present class of problems is the unusually fast convergence of the solution series. Table 1 presents the first 14 of the positive and negative eigenvalues with their corresponding expansion coefficient coefficients for $Pe = 5$, demonstrating the very fast decrease of the expansion coefficients. Thus, 4–6 terms will suffice for the temperature computation even at $\zeta = 0$ with an accuracy better than 0.05%, while for all practical purposes the first one or two terms of the series solution will be adequate. It should be perhaps also noted that the convergence is even faster for higher Peclet numbers than 5;

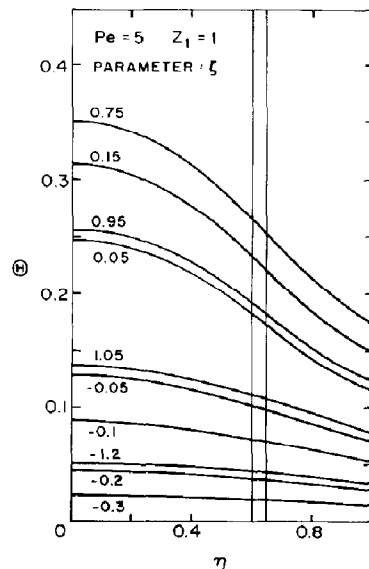


Fig. 3. Radial profiles of dimensionless temperature for various axial distances for the problem of section 3.1. $Pe = 5, Z_1 = 1, b = 0.65, d = 0.05, B = 20, k = 1$ and $B_c = 1$.

Table 1. Eigenvalues and their corresponding expansion coefficients for the problem of Section 3.1. $Pe = 5$, $b = 0.65$, $d = 0.05$, $B = 20$, $k = 1$ and $B_e = 1$.

Pe = 5				
j	$(\lambda_j^+)^{1/2}$	$m_j^+ / (\lambda_j^+ \ \phi_j^+\ ^2)$	$(-\lambda_j^-)^{1/2}$	$m_j^- / (\lambda_j^- \ \phi_j^-\ ^2)$
1	2.5759012	-.1670089	2.4763893	-.1736186
2	4.5521274	-.0247721	4.4995815	-.0274645
3	6.0824184	.0042623	6.0673884	.0041780
4	7.3140995	.0012359	7.2847805	.0013283
5	8.2099115	-.0013995	8.1913201	-.0014110
6	9.3178284	.0002265	9.3037891	.0001964
7	10.0208288	.0003381	9.9924259	.0003559
8	10.7846827	-.0002955	10.7766330	-.0002901
9	11.6908093	-.0000169	11.6726319	-.0000280
10	12.1174134	.0001535	12.0978628	.0001600
11	12.9152035	-.0000858	12.9106073	-.0000828
12	13.5896848	-.0000339	13.5638377	-.0000359
13	13.9653441	.0000774	13.9576808	.0000772
14	14.7626212	-.0000276	14.7580684	-.0000255

indeed the lower the Peclet number the slower the convergence of the series solution, as it was found with the single-stream Graetz problems, as well[1, 2]. In conclusion, the computational effort required for the present problems is significantly less even than that of the single-stream Graetz problems[1-4].

Figures 3-6 present radial temperature profiles for the problem of Section 3.1 with $b = 0.65$, $d = 0.05$, $B = 20$, $B_e = 1$, $k = 1$ with $Pe = 5$ and $Z_1 = 1$ and 2 for Figs. 3 and 4, respectively, $Pe = 40$ and $Z_1 = 0.25$ for Fig 5 and $Pe = 100$ and $Z_1 = 0.1$ for Fig. 6.

Figures 3 and 4 show the effect of the length of the heated section on the temperature profile development. Unlike the single-stream Graetz problems[1-4] the effect of axial heat conduction in the fluid cannot be ignored even for Peclet numbers larger than 40-50. Indeed, in view of the dedimensionalization process, the axial distances of Fig. 6 correspond to those of Fig. 5, and thus a comparison of the two figures shows that axial heat conduction in the fluid cannot be ignored even for $Pe = 100$. In fact, it appears that Peclet numbers significantly higher than 100 will be required for axial heat conduction in the fluid to become insignificant. The imposition of equality between the ambient and inlet coolant temperatures may seem to be an unreasonable constraint from a practical viewpoint. In this connection, the problem which is depicted in Fig. 7 overcomes this objection in the most suitable manner by allowing for a boundary discontinuity in the ambient temperature (which is naturally dealt with in the formalism) at some $z = -l_2$, where the temperature jumps from T_2 to T_1 . The section $-\infty < z < -l_2$ has an ambient temperature of T_2 , which automatically forces the fluid to enter at $-\infty$ with an inlet temperature of T_2 . The section $-l_2 < z < \infty$ has a uniform ambient temperature of T_1 , which allows the coolant to eventually assume the value T_1 at $z = \infty$. The alternative to this procedure is to insist on a *finite* axial domain, for which the formalism presented cannot provide an analy-

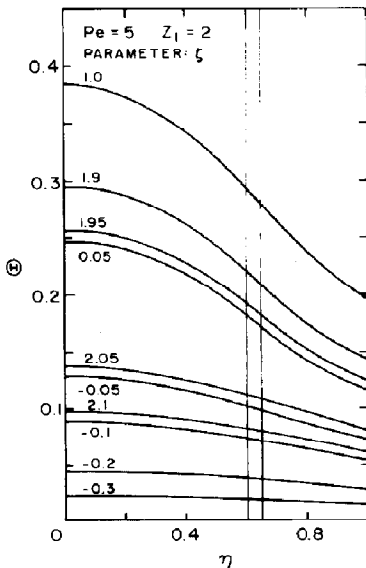


Fig. 4. Radial profiles of dimensionless temperature for various axial distances for the problem of section 3.1. $Pe = 5$, $Z_1 = 2$, $b = 0.65$, $d = 0.05$, $B = 20$, $k = 1$ and $B_e = 1$.

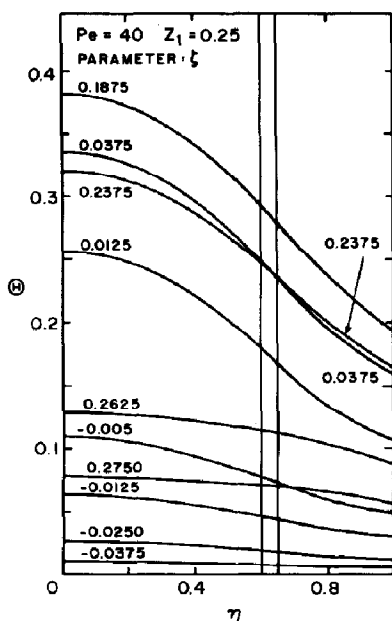


Fig. 5. Radial profiles of dimensionless temperature for various axial distances for the problem of Section 3.1. $Pe = 40$, $Z_1 = 0.25$, $b = 0.65$, $d = 0.05$, $B = 20$, $k = 1$ and $B_e = 1$.

tical solution. This disadvantage must however be weighed with the difficulty of identifying a boundary condition at the section of fluid entry. Indeed this is the altogether familiar dilemma that the modeller must face in dealing with situations where diffusion (or dispersion) is important in the flow direction.

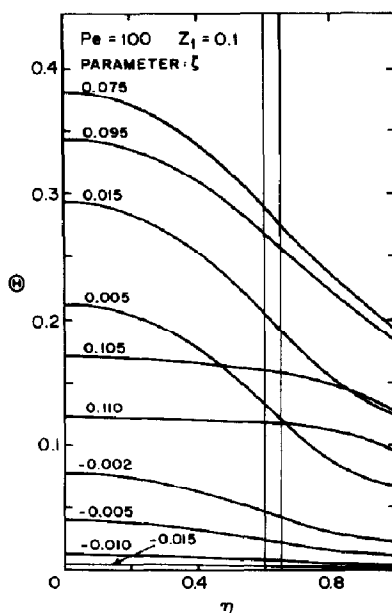


Fig. 6. Radial profiles of dimensionless temperature for various axial distances for the problem of Section 3.1. $Pe = 100$, $Z_1 = 0.1$, $b = 0.65$, $d = 0.05$, $B = 20$, $k = 1$ and $B_e = 1$.

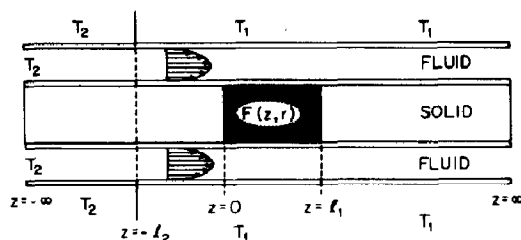


Fig. 7. A conjugated solid-fluid problem proposed to resolve the requirement of an entry fluid temperature different from the ambient temperature.

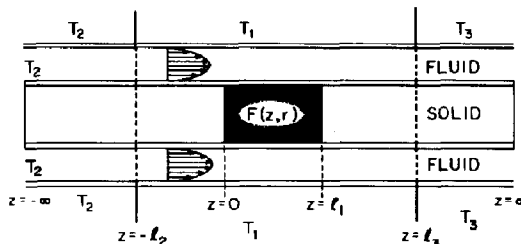


Fig. 8. A conjugated solid-fluid problem proposed to resolve the requirements of both the entry and exit fluid temperatures being different from the ambient temperature.

The foregoing considerations also hold for the downstream end where the exit temperature of the fluid at ∞ (see Fig. 8) becomes T_3 by forging a second discontinuity in the ambient temperature at some $z = l_3$.

6. CONCLUDING REMARKS

The analysis presented has shown how conjugated Graetz problems may be investigated by utilizing a single differential expression for the energy equation, irrespective of the number of phases and streams involved. In particular, we have demonstrated how solid-fluid problems may be solved simply and efficiently without resorting to the solution of their equivalent integral equations. It appears also that problems with one solid phase possess rapidly converging series solutions.

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NOTATION

- A a constant ($\equiv Pe_2/Pe_1$)
- A_1 a constant, eqn (77)
- a constant, eqn (8)
- B Biot number for the inner wall of the double tube ($\equiv hR/k_1$), eqn (12)
- B_e Biot number for the outer wall of the double tube ($\equiv h_e R/k_2$), eqn (11)
- B_j a constant, eqn (76)
- b dimensionless small radius of the annular space, Fig. 1

- C_b, C_j^+, C_j^- integration constants corresponding to λ_j, λ_j^+ and λ_j^- respectively, eqns (60)–(62)
 c_p, c_{pi} specific heats; $i = 1$ for the tubular fluid and $i = 2$ for the annular fluid
 D domain of an operator
 d dimensionless r_0 ($\equiv r_0/R$)
 E, E_1 the vectors of eqns (56) and (73), respectively
 F heat source function, Figs. 2, 7 and 8
 F, F_1 the solution vectors in \mathcal{H}
 f dimensionless F , eqn (54); or a function of η in \mathcal{H}_1 , Appendix 1
 f any three component vector in \mathcal{H}
 g a function of ζ and η , eqn (57), or a function of η in \mathcal{H}_2 , Appendix 1
 $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces, Appendix 1
 h heat transfer coefficient for the resistance of the inner wall ($\equiv k_2/(r_1 \ln(b/b-d))$), eqn (4)
 h_e heat transfer coefficient for the resistance of the outer wall, eqn (3)
 h_i a function of ζ , eqn (59)
 K_1, K_2 the constants of eqns (79) and (80), respectively
 k $\equiv (k_2/k_1) \cdot b/(b-d)$, eqn (13)
 k_1, k_2 conductivities of fluids 1 and 2, respectively
 k_i conductivity of the inner wall
 k^* fluid-conductivity (function of r), identified below eqn (5)
 L, L_1 the linear, differential operators of eqns (35) and (73), respectively
 l_1 the length of the heat source, Figs. 2, 7 and 8
 l_2, l_3 characteristic lengths, Figs. 7 and 8
 M the differential operator of eqn (9)
 m_i a constant, eqn (66a)
 Pe Peclet number ($\equiv Pe_2$)
 Pe_i Peclet number of fluid i ($\equiv V_i R \rho_i c_{pi} / k_i$); $i = 1$ for the tubular fluid and $i = 2$ for the annular fluid
 R the radius of the outer tube of the double pipe
 \mathcal{R} the set of real numbers
 r radial variable
 r_0 thickness of the inner wall of the double tube, Fig. 1
 r_1 radius of the inner tube of the double pipe, Fig. 1
 r_2 width of the annulus of the double pipe, Fig. 1
 S axial energy flow function, eqn (21)
 S_1, S_2 axial-energy flow functions for fluids 1 and 2, eqns (15) and (16), respectively
 T temperature
 T_1, T_2 characteristic temperatures, Figs. 2, 7 and 8
 T_D characteristic temperature; for the problems of section 3 $T_D = T_2$
 V_1, V_2 characteristic velocities corresponding to fluids 1 and 2, eqns (6) and (7), respectively
 v dimensionless v_z , eqn (14)
 v_z fluid velocity function for the whole double tube, identified below eqn (5)
 v_1, v_2 dimensionless v_{z1} and v_{z2} , respectively, eqn (14)
 v_{z1}, v_{z2} velocities of fluids 1 and 2, respectively, eqns (6) and (7), respectively
 X the function of ϕ and ψ defined by eqn (41)
 Z_1 dimensionless l_1
 z axial variable
- Greek symbols**
- γ a constant defined by eqn (28)
 δ a constant; +1 for co-current and -1 for counter-current conjugated problems, eqn (7)
 ζ dimensionless z ($\equiv z/RPe$)
 η dimensionless r ($\equiv r/R$)
 Θ dimensionless T ($\equiv (T - T_2)/T_D$)
 Θ_∞ the temperature of eqn (2.1) of Appendix 2
 $\lambda_1, \lambda_1^+, \lambda_1^-$ eigenvalues, positive and negative eigenvalues, respectively
 ρ fluid density (function of r), identified below eqn (5)
 ρ_1, ρ_2 densities of fluids 1 and 2, respectively
 Σ dimensionless flow function S , eqn (25)
 Σ_1, Σ_2 dimensionless S_1 and S_2 , respectively, eqns (23) and (24)
 Φ a function of η , eqn (78)
 ϕ any three component vector in \mathcal{H}
 $\phi_1, \phi_2, \phi_3, \phi_4$ the first, second, third and fourth components, respectively, of ϕ
 $\phi_j, \phi_j^+, \phi_j^-$ eigenvectors corresponding to λ_j, λ_j^+ and λ_j^- , respectively
 ϕ_{j1}, ϕ_{j2} the first and second components of ϕ_j^+ , respectively
 Ψ a function of η , eqn (68)
 ψ any three components vector in \mathcal{H}
 $\psi_1, \psi_2, \psi_3, \psi_4$ the first, second, third and fourth components respectively of ψ
 $\langle \cdot, \cdot \rangle$ the inner product of eqn (40)
 $\| \cdot \|$ the norm that corresponds to $\langle \cdot, \cdot \rangle$

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APPENDIX 1

An auxiliary proof: symmetric operator L

Let I be the union of the intervals $[0, (b-d)]$ and $((b, 1])$. Consider now the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{R} \oplus \mathcal{R}$ where \mathcal{H}_1 is the space of all functions $f(\eta)$ in I such that the Lebesgue integral

$$\int_I \eta f^2(\eta) d\eta < \infty.$$

\mathcal{H}_2 is the space of all functions $g(\eta)$ defined on I such that the Lebesgue integral

$$\int_I \frac{1}{\eta} g^2(\eta) d\eta < \infty.$$

and \mathcal{R} is the set of real numbers. Clearly, \mathcal{H} is a Hilbert space with inner product between two vectors $\phi, \psi \in \mathcal{H}$ given by eqn (40) where ϕ , and ψ are defined directly above that equation. Let ϕ and $\psi \in D(L)$ where $D(L)$ is given by eqn (42); then we have

$$\begin{aligned} \langle L\phi, \psi \rangle - \langle \phi, L\psi \rangle = & 2 \left[\int_0^{b-d} (\psi_2 \phi_1 - \psi_1 \phi_2) d\eta + \gamma \int_b^1 ((\psi_2 \right. \\ & + \gamma \int_b^1 (\psi_2 \phi_1 - \psi_1 \phi_2) d\eta - \{(\phi_1(b-d) - \phi_1(b))\psi_2(b-d) \\ & - \{\psi_1(b-d) - \psi_1(b)\}\phi_2(b-d)) - \gamma\phi_1(1)\psi_2(1) + \gamma\phi_2(1)\psi_1(1) \Big] = \\ & 2[\psi_2(b-d)\phi_1(b-d) - \psi_1(b-d)\phi_2(b-d) \\ & - \psi_2(0)\phi_1(0) + \psi_1(0)\phi_2(0) + \\ & + \gamma\psi_2(1)\phi_1(1) - \gamma\psi_1(1)\phi_2(1) - \gamma\psi_2(b)\phi_1(b) + \gamma\psi_1(b)\phi_2(b) \\ & - \phi_1(b-d)\psi_2(b-d) + \phi_1(b)\psi_2(b-d) + \psi_1(b-d)\phi_2(b-d) \\ & - \psi_1(b)\phi_2(b-d) - \gamma\phi_1(1)\psi_2(1) + \gamma\phi_2(1)\psi_1(1)] = 0 \end{aligned}$$

that is, L with $D(L)$ is a symmetric operator in \mathcal{H} .

APPENDIX 2

From eqn (67b) and for $Z_1 \rightarrow \infty$ we obtain,

$$\Theta_\infty(\zeta, \eta) = \sum_{j=1}^{\infty} \frac{m_j \phi_{1j}(\eta)}{\lambda_j \|\phi_j\|^2} = \Psi(\eta). \tag{2.1}$$

Substituting $\Psi(\eta)$ into the energy eqn (54) and in view of (63) we obtain,

$$-\frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{d\Psi}{d\eta} \right) = \begin{cases} 1 & \eta \in [0, b-d] \\ 0 & \eta \in [b, 1] \end{cases} \tag{2.2a}$$

$$\tag{2.2b}$$

subject to boundary conditions (10)–(13). Thus, we may solve the above boundary value problems to obtain the solution given by eqn (68).